

Entanglement without Tomography: A Rankwise Study

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Experimental determination of entanglement is important not only to characterize the state and use it in quantum information, but also in understanding complicated phenomena such as phase transitions. In this paper we show that in many cases, it is possible to determine entanglement of a two qubit state, as represented by concurrence, with a few measurements, *most of which are local*. In particular, rank one and rank 2 states need exclusively local measurements while rank 3 states need just one measurement of correlations. Only the rank four states are shown to require a more detailed tomography. The analysis also sheds light on the other measure, nonseparability since it is a lower bound on concurrence.

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I. INTRODUCTION

The purpose of this paper is to examine the possibility of determining entanglement of two qubit states without complete tomography. We look for minimal sets of observables that can specify the entanglement of a given state. An answer to this question would save an experimentalist of performing complete tomography – which is by no means a trivial task. It will also bring out the invariant character of entanglement with respect to a host of transformations, especially local transformations. Since the interest in entanglement has now transcended the domain of quantum information and is being used in understanding varied physical phenomena, ranging from black hole thermodynamics to phase transitions, it is all the more pertinent to identify sets of observables which can decisively yield information on entanglement. Finally, not all observables can be measured with equal ease. For instance, measuring local observables is much easier than measuring correlations, and it would be good to employ easily measurable observables.

This topic is not new and has been addressed earlier in a variety of contexts. Experimentally, tomography has been the natural route to determine entanglement. Tomographic techniques have been well developed and applied for different qubit systems, viz. superconducting qubits [1, 2], entangled photons [3]. The techniques employed depend on the system considered, and are well summarized recently in [4]. Tomography involves multiple measurements on identical copies and constructing density operator [5–9]. We may also mention, as examples, tomography of photons entangled in high dimensions [5]. A complete state tomography requires measurement of a complete set of observables, which is essentially equal to number of elements in density operator, hence the complexity scales exponentially with di-

mensionality of quantum system [5, 9]. Indeed, some cases of tomography have been reported, which require fewer measurements, but these cases are either limited to approximation schemes [10], or rather very specific quantum states [11]. We refer to [12–18] which employ such specific states. Theoretical investigations have been conducted to reconstruct density matrix efficiently using numerical methods [19].

The strategy employed in this paper is to analyze the states rankwise. Similar approaches have been employed successfully earlier [20–25]; however, there are differences between those works and ours in several aspects. Some of them employ numerical or nondeterministic approach [23, 24]. The very interesting result in [20] that very few measurements are required to determine entanglement of formation requires joint measurements of subsystems while our approach focusses on maximizing the number of local measurements. Our demonstrations are constructive and we employ the full machinery of invariance under local transformations. Earlier, Yang et al. [26] have discussed two qubit entanglement in terms of invariants of three qubit systems. Such approach presumes two qubit state (2QS) to be a daughter state of parent three qubit system and hence the states under considerations are not very general. A general discussion of 2QS entanglement should be expressed in terms of 2QS invariants. In some way, our methods are more general because most of our findings are applicable to all 2QS of that rank.

II. ENTANGLEMENT MEASURES

We first need to specify a measure of entanglement before identifying the relevant observables. We consider the pure states first, partly for completeness and partly to provide the setting.

A. Pure states

Let $|\Psi\rangle$ be a state of a bipartite system with subsystems A, B . Its entanglement, $E(\Psi)$ could be specified in many ways such as violation of Bell inequalities, or the entropy of one of the reduced states. Fortunately, all such characterizations are equivalent in the sense that they are relative monotones of each other. In this paper, We choose pure state concurrence as our measure since it easily generalizes to the 2QS. It can be conveniently expressed as $C(\Psi) = 2|\langle\tilde{\Psi}|\Psi\rangle|$ where $|\tilde{\Psi}\rangle$ is its time reversed state. More explicitly, if we employ the language of spin – which we do throughout for convenience, writing $|\Psi^{AB}\rangle = \sum_{m\mu} c_{m\mu}|m, \mu\rangle$; $m, \mu = \pm 1/2$, the time reversed state is given by $|\tilde{\Psi}\rangle = \sum_{m\mu} (-1)^{m+\mu} c_{-m, -\mu}|m, \mu\rangle$. Thus the expression for pure state concurrence is given by

$$C(\Psi) = 2|c_{1/2, 1/2}c_{-1/2, -1/2} - c_{1/2, -1/2}c_{-1/2, 1/2}| \quad (1)$$

The factor 2 in the above expression bounds the measure in the interval $[0, 1]$.

Before we proceed further, it is pertinent to note that any measure of entanglement is required to be invariant under local transformations, $SU(2) \times SU(2)$ which is a proper subgroup of $SU(4)$. Since there are only two independent local invariants for a pure state, its norm being one, it follows that all measures are guaranteed to be equivalent.

B. Mixed state measures

The case with mixed states is more involved for several reasons. (i) It is first necessary to fix the measure of entanglement that we wish to employ for mixed states. For, unlike the pure case, there is no unique measure of entanglement. Several measures that have been proposed [27–30] are all inequivalent to each other; they are not relative monotones of each other. Indeed, a mixed state can have as many as nine local invariants, and any measure of entanglement is a complicated function of these invariants. (ii) Even if one imagines that nine measurements should, therefore, suffice, it may not be the case because experimental techniques are generally geared to measure single particle observables and correlations. Even theoretically, expressing a measure of entanglement in terms of a standard set of invariants is a tedious task. Consequently, it is worthwhile looking for sets of optimal measurements which can determine entanglement maximally.

The choice of observables naturally depends on the measure of entanglement. In this paper we employ concurrence which naturally generalizes pure state concur-

rence [28]. It has a rather involved algebraic expression given by [31]

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \quad (2)$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the square roots of eigenvalues of $\rho\tilde{\rho}$ in decreasing order and $\tilde{\rho}$, the spin flipped density operator is essentially the time reversed state of ρ .

III. LOCAL INVARIANTS

Let the state be written in its standard form

$$\rho_{\Psi}^{AB} = \frac{1}{4} \left\{ 1 + \vec{\sigma}^A \cdot \vec{P} + \vec{\sigma}^B \cdot \vec{S} + \sigma_i^A \sigma_j^B \Pi_{ij} \right\}, \quad (3)$$

It is convenient to further define, in addition,

$$\begin{aligned} A_i &= \Pi_{ij} P_j; & B_i &= S_j \Pi_{ji} \\ \vec{\alpha} &= \vec{P} \times \vec{A}; & \vec{\beta} &= \vec{S} \times \vec{B} \\ T_{ij} &= \Pi_{ik} \Pi_{jk} \end{aligned} \quad (4)$$

A set of independent invariants can then be conveniently listed as follows:

$$\begin{aligned} I_1 &= \vec{P}^2; & I_2 &= \vec{S}^2 \\ I_4 &= \vec{A}^2; & I_3 &= \vec{P} \cdot \vec{A} = \vec{S} \cdot \vec{B} \\ I_5 &= \vec{B}^2; & I_6 &= \text{Tr} T \\ I_8 &= \text{Tr} T^2; & I_7 &= A_i \Pi_{ij} B_j \\ I_9 &= \alpha_i \Pi_{ij} \beta_j. \end{aligned} \quad (5)$$

Since entanglement is invariant under local transformations, it would appear that even in the most general case, a measurement of the above nine invariants should suffice. In fact, since concurrence is invariant for states under much larger set of transformation. For example, concurrence vanishes for a pure separable state (of rank 1), any three dimensional projection (which is of rank 3) and of course the fully unpolarized state (of rank 4) – one may even expect a further decrease the number of measurements. However, such a counting is rather too simplistic; one only need to look at the form of these invariants in Eqn.5. It would be a tall order to ask that experiments be designed to measure these invariants directly.

With this in mind we approach the problem by examining specific classes of states with increasing degree of complexity. We choose the rank of the state, $R(\rho)$ to characterize the state. Finer distinctions will be made for each rank as and when possible.

IV. ENTANGLEMENT OF PURE STATES

$$(R(\rho) = 1)$$

Well known though this case is, a brief discussion would highlight several features which can be exploited for the more complicated mixed states.

A. Determination of $C(\Psi)$

The expression for pure state concurrence is given in Eqn.1. To determine the observables, it is better to rewrite it in the form given in Eqn.3. The requirement of purity, $\rho^2 = \rho$, imposes additional conditions

$$\begin{aligned} \vec{P}^2 &= \vec{S}^2 \\ 2\vec{P}^2 + TrT &= 3. \end{aligned} \quad (6)$$

Well known that these relations are, it is still noteworthy that the strength of the correlation, given by TrT , is entirely determined by the single qubit invariant, \vec{P}^2 . Thus, although entanglement depends on correlation, it can more easily be determined by performing a simpler measurement, *viz.*, of the degree of polarization, \vec{P}^2 of either of the subsystems. Indeed, concurrence is simply given by $C(\Psi) = \sqrt{1 - \vec{P}^2}$. In short, a local measurement determines the nonlocal character of the state unambiguously. One hopes that similar opportunities present themselves for mixed states as well.

We systematically extend our studies to states of increasing rank, by discussing special examples in each case.

V. RANK 2 STATES

Rank two states occur naturally as descendants over a three qubit pure state state. Physical realizations abound. The final states in the celebrated beta decay, $n \rightarrow p + e^- + \bar{\nu}$ is one such example. Monogamy relations [32] make the concurrences of the three 2-qubit subsystems mutually constraining.

Let us begin with a convenient representation of a rank 2 state in its eigenbasis:

$$\rho_2 = \nu |\chi\rangle\langle\chi| + (1 - \nu) |\chi_\perp\rangle\langle\chi_\perp|. \quad (7)$$

It is further convenient to employ the Schmidt basis to expand $|\chi\rangle$, and employ the residual freedom in local operations, together with the overall phase, to simplify the form of $|\chi_\perp\rangle$. Thus,

$$\begin{aligned} |\chi\rangle &= \cos\alpha|00\rangle + \sin\alpha|11\rangle \\ |\chi_\perp\rangle &= \cos\eta\{\sin\alpha|00\rangle - \cos\alpha|11\rangle\} \\ &\quad + \sin\eta\{\sin\beta\exp(-i\gamma)|01\rangle + \cos\beta|10\rangle\} \end{aligned} \quad (8)$$

where $0 \leq \alpha, \beta, \eta \leq \pi/2$ and $0 \leq \gamma \leq 2\pi$. Thus the manifold of inequivalent rank two states (under LO) is characterized by a family of four parameters, $\nu, \alpha, \beta, \gamma$. Furthermore, concurrence is a function of only local invariants. It remains to relate them to easily measurable observables.

Claim I: Concurrence of a rank 2 state is completely determined by local measurements.

To show this, we prove a stronger result:

Claim I': A rank 2 state is completely determined by its daughter states, upto local transformations.

The proof is by explicit construction. The local observables can be read off from Eqn.8 to be

$$\begin{aligned} P_x &= 2(1 - \nu) \cos\eta \sin\eta \{ \cos\alpha \cos\beta + \sin\alpha \sin\beta \cos\gamma \} \\ P_y &= -2(1 - \nu) \cos\eta \sin\eta \sin\alpha \sin\beta \sin\gamma \\ P_z &= \nu \cos 2\alpha + (1 - \nu) \{ -\cos^2\eta \cos 2\alpha \\ &\quad + \sin^2\eta \cos 2\beta \} \end{aligned} \quad (9)$$

Similarly the expression for \vec{S} can be read off as

$$\begin{aligned} S_x &= 2(1 - \nu) \cos\eta \sin\eta \{ -\sin\alpha \cos\beta + \sin\alpha \sin\beta \cos\gamma \} \\ S_y &= -2(1 - \nu) \cos\eta \sin\eta \cos\alpha \sin\beta \sin\gamma \\ S_z &= \nu \cos 2\alpha - (1 - \nu) \{ \cos^2\eta \cos 2\alpha \\ &\quad + \sin^2\eta \sin 2\beta \} \end{aligned} \quad (10)$$

Note that $\sin\alpha$ and $\cos\alpha$ determine each other as is the case with $\sin\beta$, $\cos\beta$ and also $\cos\eta$, $\sin\eta$. However, one needs to measure both $\cos\gamma$ and $\sin\gamma$ for determining γ . Thus six measurements are required to determine the state, though the manifold is itself five dimensional.

It is easy to see from Eqns. 9,10 that the two single qubit polarizations determine the state completely. We outline the sequential steps below:

1. The ratio P_y/S_y determines α unambiguously.
2. We next note that the ratios $P_x/S_x \equiv R_1(\beta, \gamma)$ and $P_y/P_x \equiv R_2(\beta, \gamma)$ together determine β, γ . There would still be a discrete ambiguity in γ .
3. Combining them with $P_z \pm S_z$, ν, β are also determined.
4. Finally, the discrete ambiguity in the value of γ can be resolved by employing either P_y or S_y .

We have thereby proved that a complete tomography of a rank 2 state, upto its local equivalents, can be accomplished entirely by single qubit observables¹.

Concurrence is a function of local invariants, and they are simply given by I_1, I_2 in this case. We have thus shown that a measurement of two single qubit invariants is necessary, and sufficient to determine $C(\rho)$ whenever it is of rank 2. In the limiting case when $\nu = 0, 1$, the two invariants are equal, thereby reducing to the pure case discussed in the previous section.

To bring out vividly the result proved, we consider several examples which are based on another equivalent representation of rank 2 states, as an incoherent superposition of a separable and a pure state:

$$\rho_2 = \lambda \rho_s + (1 - \lambda) |\psi\rangle\langle\psi| \quad (11)$$

where the rank of the separable state, $R(\rho_s) \leq 2$.

Accordingly, we consider two cases, $R(\rho_s) = 1, 2$ separately. We first discuss the latter case, and then go on to show the exceptional nature of rank-1 states.

A. $R(\rho_s) = 2$

The most general separable state, ρ_s is a mixture of two separable and mutually orthogonal states:

$$\rho_s = \mu |\chi_1\rangle\langle\chi_1| + (1 - \mu) |\chi_2\rangle\langle\chi_2| \quad (12)$$

Employing local transformations, we can always write $|\chi_1\rangle = |00\rangle$ and $|\chi_2\rangle = a|10\rangle + b|11\rangle$ where a, b can be taken to be real and nonnegative. Note that $|\psi\rangle$ necessarily lies in the plane spanned by $|\chi_1\rangle$ and $|\chi_2\rangle$, and may be written as

$$|\psi\rangle = \cos\theta |00\rangle + \exp(i\beta) \sin\theta (a|10\rangle + b|11\rangle) \quad (13)$$

Combining Eqns. 11, 12 and 13 we get the canonical form of the state. The algebra to determine concurrence is rather involved but straightforward. The basic steps are:

1. Construct the spin flip density operator $\tilde{\rho}_2$ and compute $\rho_2 \tilde{\rho}_2$.
2. Solving for eigenvalues of $\rho_2 \tilde{\rho}_2$ becomes simpler due to the fact that two of the eigenvalues vanish.

3. Once eigenvalues are known it's straightforward to write concurrence as a function of elements of the density operator and further as a function of local invariants.

The final expression is quite neat, and is given by

$$C = \max(\sqrt{1 - I_1}, \sqrt{1 - I_2}) \quad (14)$$

The above expression is similar to the pure state case, but while pure state entanglement can be determined by measurement on only one of the qubit, entanglement of this case requires measurement on both qubits.

B. $R(\rho_s) = 1$

We use this case to give an example of a rank 2 state which is an exception to the theorem proved. For that, we consider a special case when the two one dimensional projections are orthogonal to each other. Again, using the freedom under local transformations, we may employ the forms

$$\begin{aligned} \rho_s &= |00\rangle\langle 00| \\ |\psi\rangle &= r_1 |01\rangle + c |10\rangle + r_2 |11\rangle \\ \rho_2 &\equiv \lambda |00\rangle\langle 00| + (1 - \lambda) |\psi\rangle\langle\psi| \end{aligned} \quad (15)$$

where $r_{1,2}$ are real and nonnegative.

The evaluation of concurrence is again straight forward and all the steps involved are similar to the last case. It has a very simple form given by

$$C = (1 - \lambda) 2r_1 |c| \equiv (1 - \lambda) C(|\psi\rangle) \quad (16)$$

Simple though the expression is, it is clear that single qubit invariants are not sufficient since the determination of the eigenvalue λ requires a knowledge of tensor correlations as well. Such states necessitate a determination of an additional quantity such as $Tr \rho^2$ which is a measure of mixedness of ρ_2 : $Tr(\rho_2^2) = 2\lambda^2 - 2\lambda + 1$.

1. A physical example

We show how the preceding analysis is applicable to a physical system involving quantum phase transition, which has been studied by Wu et al[33]. They have studied quantum phase transition of a Frustrated two-leg spin-1/2 ladder system and connected the nonanalyticity in the energies directly to bipartite entanglement, which gets manifested in both concurrence and negativity. And

¹ There could be some exceptions for some values of parameters, where the equations become degenerate. But they are clearly a set of measure zero. We discuss one such example in the following subsection.

it follows from our analysis that measuring concurrence for this system is relatively easy! The density operator of the rung is described by statistical mixture of singlets and triplets, where ground state being the mixture with equal mixture of singlets and triplets. It naturally has the form given in Eqn.15 with $r_2 = 0$, $r_1 = |c| = \frac{1}{\sqrt{2}}$. It is easy to see that concurrence for this state is simply given by

$$C = 2(1 - \lambda)|b||c| \equiv 1 - \rho_{11} \quad (17)$$

in the basis labeled by a convenient measurement basis. This is one of the few examples where a single occupation probability determines entanglement completely.

2. Two dimensional projections

To illustrate the above remark, we consider the special case when the state is a two dimensional projection. This corresponds to $\lambda = \frac{1}{2}$ in Eqn.15. It is easy to verify that concurrence can be expressed in terms of the single qubit invariants as

$$C = \sqrt{\frac{1 - I_1 - I_2}{2}} - \sqrt{\frac{(I_1 - I_2)^2}{4} - \frac{I_1 + I_2}{2} + \frac{1}{4}} \quad (18)$$

VI. RANK-3 STATES

Rank three states are considerably more complex, and as we show below, measurement of correlations is inevitable in most situations. Yet we may ask how far one can take the program of minimal measurements through. To examine that, let us write the state in its eigen basis:

$$\rho_3 = \nu_1 |\chi_1\rangle\langle\chi_1| + \nu_2 |\chi_2\rangle\langle\chi_2| + (1 - \nu_1 - \nu_2) |\chi_3\rangle\langle\chi_3| \quad (19)$$

where we have arranged the eigenvalues ν_i in the non-decreasing order. The eigenstates χ_i may be brought to their canonical form under local transformations. We may choose the first two eigenstates to be as given in Eqn.8 and write the third eigenstate as

$$\begin{aligned} |\chi_1\rangle &= \cos \alpha |00\rangle + \sin \alpha |11\rangle \\ |\chi_2\rangle &= \cos \eta \{ \sin \alpha |00\rangle - \cos \alpha |11\rangle \} \\ &\quad + \sin \eta \{ \sin \beta \exp(-i\gamma) |01\rangle + \cos \beta |10\rangle \} \\ |\chi_3\rangle &= \cos \xi \{ \sin \alpha |00\rangle - \cos \alpha |11\rangle \} \\ &\quad + \sin \xi \{ \sin \theta \exp(-i\phi_1) |01\rangle + \cos \theta \exp(-i\phi_2) |10\rangle \} \end{aligned} \quad (20)$$

The orthogonality conditions impose two further constraints on the angles. Thus the state, in its canonical form, has eight independent parameters.

An immediate corollary is that local measurements are not sufficient to determine concurrence and must be necessarily supplemented with measurements of correlations. The same feature is inherited by rank 4 states, since they would be characterized by the maximal number of parameters, viz, six, in their canonical form². While there could be special cases where local measurements may suffice, we consider, instead an alternative problem: that of extraction of maximal information on concurrence with insufficient number of measurements. That is necessarily of the nature of placing bounds, and is closer in spirit to the approach of Plenio [34, 35].

VII. BOUNDS ON CONCURRENCE

A. Rank 3 states

With a slight rearrangement we can write a rank 3 state as $\rho_3 = \lambda \Pi_3 + (1 - \lambda) \rho_2$ where Π_3 is a three dimensional projection and $R(\rho_2) = 2$. Further employing the resolution in Eqn.11 for ρ_2 , we get

$$\rho_3 = \lambda \Pi_3 + (1 - \lambda) [\mu \rho_{sep} + (1 - \mu) |\psi\rangle\langle\psi|] \quad (21)$$

The only contribution to entanglement of the state described by ρ_3 in Eqn.21 is from the pure state component, $|\psi\rangle$ since the other terms are concurrence free. Thus, we have a rather weak bound

$$C(\rho_3) \leq (1 - \lambda)(1 - \mu)C(\psi) \leq (1 - \lambda)C(\psi) \quad (22)$$

The pure state, $|\psi\rangle$ is constrained to lie in the three dimensional space \mathcal{H}_3 projected by Π_3 .

$$\begin{aligned} |\psi\rangle &= \cos \theta (a |01\rangle + b |10\rangle) \\ &\quad + \sin \theta \cos \phi |00\rangle + \sin \theta \sin \phi |11\rangle \end{aligned} \quad (23)$$

An interesting question is to find the upper limit on λ for the projection Π_3 for a state to be entangled. That happens when the weight associated with ρ_{sep} vanishes, and the pure component is fully entangled. The phases are not of much concern for this maximally entangled state, and thus,

$$\rho_3^{max} = \begin{bmatrix} \frac{\lambda}{3} & 0 & 0 & 0 \\ 0 & (1 - \frac{2\lambda}{3})a^2 & (1 - \frac{2\lambda}{3})ab & 0 \\ 0 & (1 - \frac{2\lambda}{3})ab & (1 - \frac{2\lambda}{3})b^2 & 0 \\ 0 & 0 & 0 & \frac{\lambda}{3} \end{bmatrix} \quad (24)$$

² Six out of the fifteen parameters can be transformed away by local transformations

for which, the concurrence is given by

$$C(\rho_3^{max}) = 2(1 - \frac{2\lambda}{3})ab - \frac{2\lambda}{3} \quad (25)$$

which implies that for entanglement

$$\lambda < \frac{3ab}{1 + 2ab} \quad (26)$$

The R.H.S. of above expression attains its maxima at $ab = \frac{1}{2}$, therefore, the state is entangled only if $\lambda < \frac{3}{4}$.

Theorem-Any 2-qubit, bipartite quantum state described by a rank-3 density operator is entangled only if the mixture has less than $\frac{3}{4}$ of the separable part.

These bounds on entanglement are compatible with the results that have been previously discussed by Wunderlich, Plenio[34] and Audenaert, Plenio[35] in terms of local measurement of correlation observables. It is straightforward to get the bound of 26 in terms of observables, since $\lambda = \frac{3}{4}(\langle s_z p_z \rangle + 1)$, for ρ_3^{max} .

B. The rank-4 case

In this most general case, we have absolutely no prior information on the state, and a complete tomography is inescapable. We may yet ask, as with rank 3 states, on the kinds of inferences that we may draw from partial information. To do so, we arrange the eigenexpansion

$$\rho_4 = \lambda_1 \mathbb{I}_4 + \lambda_2 \pi_3 + (1 - \lambda_1 - \lambda_2) \rho_2 \quad (27)$$

Using the familiar resolution $\rho_2 = \nu \rho_{sep} + (1 - \nu) |\psi\rangle\langle\psi|$, we see that the only contribution to the entanglement of ρ_4 is from the pure state described by $|\psi\rangle$. The concurrence of ρ_4 is bounded by the relation

$$\begin{aligned} C(\rho_4) &\leq (1 - \lambda_1 - \lambda_2)(1 - \mu)C(\psi) \\ &\leq (1 - \lambda_1 - \lambda_2)C(\psi) \end{aligned} \quad (28)$$

Thus the maximal value of concurrence for a rank-4 state with $\lambda_{1,2}$ fixed is, Thus, if the two smallest eigenvalues are known, ρ_4 is maximally entangled when, contribution of ρ_{sep} in the mixture is zero and $|\psi\rangle$ is maximally entangled, i.e. $|\psi\rangle$ is a Bell state. In that case, we find that

$$C(\rho_4^{max}) = \frac{\lambda_2}{3}ab + \frac{1 - \lambda_1 - \lambda_2}{2} - \frac{\lambda_1}{4} - \frac{\lambda_2}{3} \quad (29)$$

The above equations along with maximally entangled condition $ab = \frac{1}{2}$, impose the constraint

$$9\lambda_1 + 8\lambda_2 < 6 \quad (30)$$

In addition to the normalization constraint $\lambda_1 + \lambda_2 \leq 1$.

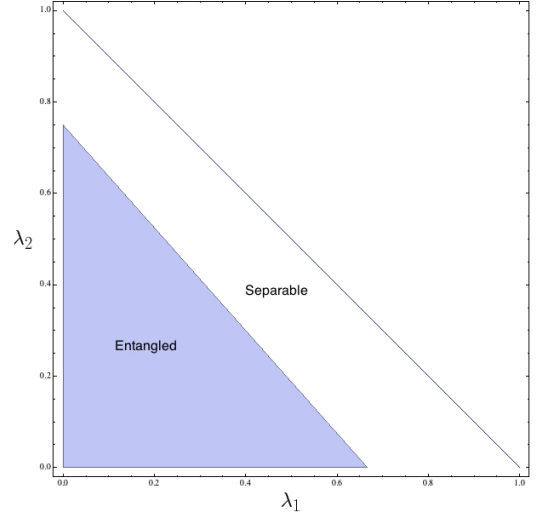


FIG. 1: Entanglement-Separability region for the rank-4 density matrix. The axes λ_1 and λ_2 represent fraction identity (completely mixed part) and rank-3 operator in the entanglement signal, respectively.

We get back the rank-3 case for $\lambda_1 = 0$, which is consistent with the result.

Theorem 4- Any 2-qubit, bipartite quantum state is entangled only if λ_1 and λ_2 as discussed above, which represent fractions of separable part in quantum state follow the relations:

$$\begin{aligned} 9\lambda_1 + 8\lambda_2 &< 6 \\ \lambda_1 + \lambda_2 &\leq 1 \end{aligned} \quad (31)$$

Finally, we conclude this section by exhibiting how correlation measurements determine the eigenvalues. Indeed, a simple exercise shows that

$$\begin{aligned} \lambda_1 &= 1 - \langle s_z p_z \rangle - 2\langle s_x p_x \rangle \\ \lambda_2 &= \frac{3}{2}(\langle s_z p_z \rangle + \langle s_x p_x \rangle) \end{aligned}$$

Above equations in combination with Eqn.31 gives bounds in terms of correlation observables for our maximal state.

VIII. APPENDIX

The expectation value of correlation measurement on ρ_3^{max} given by eqn.24 is given by,

$$\langle s_z p_z \rangle = \frac{\lambda}{3} - (1 - \frac{2\lambda}{3}) + \frac{\lambda}{3} \quad (A1)$$

This gives λ in terms of correlation measurement.

The expectation values of correlation measurements on ρ_4^{max} are given by,

$$\begin{aligned}
 \langle s_x p_x \rangle &= \frac{2\lambda_2}{3} ab + 1 - \lambda_1 - \lambda_2 \\
 &= 1 - \lambda_1 - \frac{2\lambda_2}{3} \\
 \langle s_z p_z \rangle &= \frac{\lambda_1}{2} + \frac{2\lambda_2}{3} - \frac{\lambda_2}{3} - \frac{2 - \lambda_1 - 2\lambda_2}{2} \\
 &= \lambda_1 + \frac{4\lambda_2}{3} - 1
 \end{aligned} \tag{A2}$$

Here, we have taken the maximally entangled case by substituting $ab = \frac{1}{2}$. The above equations can be solved to give λ_1 and λ_2 as functions of correlation measurements.

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